DISCRETE MATHEMATICS 9 (1974) 359–363. © North-Holland Publishing Company

# MONOTONIC TRIADS

B.G. EKE
*Department of Pure Mathematics, University of Sheffield, Sheffield, England*

Received 1 December 1973 *

## 1.

We consider sequences $X$ with $n$ terms and let $N(n, 3, X)$ denote the number of monotonic subsequences with three terms (triads) contained in $X$. It is clear that each of the sequences $[\frac{1}{2}n], [\frac{1}{2}n]-1, ..., 1, n, n-1, ..., [\frac{1}{2}n]+1$ and $[\frac{1}{2}n]+1, ..., n, 1, 2, ..., [\frac{1}{2}n]$ possesses

$$\binom{[\frac{1}{2}n]}{3}+\binom{[\frac{1}{2}(n+1)]}{3}=\begin{cases}\frac{1}{24}(n^3-6n^2+8n), & n \text{ even},\\ \frac{1}{24}(n^3-6n^2+11n-6), & n \text{ odd},\end{cases}$$

monotonic triads.

**Theorem.**

$$N(n,3,X) \geqslant \begin{cases}\frac{1}{24}(n^3-6n^2+8n), & n \text{ even}, & (1)\\ \frac{1}{24}(n^3-6n^2+11n-6), & n \text{ odd}. & (2)\end{cases}$$

*Equality is possible only for the two sequences exhibited above when $n$ is even and for four sequences when $n$ is odd, viz., the two exhibited together with $[\frac{1}{2}n]+1, [\frac{1}{2}n], ..., 1, n; n-1, ..., [\frac{1}{2}n]+2$ and $[\frac{1}{2}n]+2, [\frac{1}{2}n]+3, ..., n, 1, 2, ..., [\frac{1}{2}n]+1$.*

Burkill and Mirsky [1, p. 397] obtained (1) and we reproduce the proof for completeness. It is clear that we may assume the given sequence to be of distinct numbers and since we are only interested in order we look at permutations $S = (x_1, x_2, ..., x_n)$ of $(1, 2, ..., n)$. Write $\alpha_k$ for the number of numbers in $(x_1, x_2, ..., x_{k-1})$ which are less than $x_k$. Then the number

* Original version received 20 September 1973.

of monotonic triads with middle term $x_k$ is

$$\alpha_k(n-k-x_k+1+\alpha_k)+(k-1-\alpha_k)(x_k-1-\alpha_k) =$$

$$=(k-1)(x_k-1)-\tfrac{1}{8}(n+3-2k-2x_k)^2+2\{\alpha_k+\tfrac{1}{4}(n+3-2k-2x_k)\}^2.$$

Hence $S$ has $\Sigma_1+\Sigma_2$ monotonic triads, where

$$\Sigma_1=\sum_{k=1}^{n}\{(k-1)(x_k-1)-\tfrac{1}{8}(n+3-2k-2x_k)^2\}$$

$$=\tfrac{1}{24}(n^3-6n^2+5n), \qquad (3)$$

$$\Sigma_2=\sum_{k=1}^{n}2\{\alpha_k+\tfrac{1}{4}(n+3-2k-2x_k)\}^2.$$

When $n$ is even, $|\alpha_k+\frac{1}{4}(n+3-2k-2x_k)|\geq\frac{1}{4}$ and so $\Sigma_2\geq\frac{1}{8}n$ and (1) follows. To establish (2) we need to prove

$$\Sigma_2\geq\tfrac{1}{4}(n-1), \quad n \text{ odd}. \qquad (4)$$

## 2.

The pair $(k, x_k)$ will be called a zero pair if $\alpha_k=\frac{1}{4}(2k+2x_k-n-3)$. For other pairs $(k, x_k)$ the contribution to $\Sigma_2$ is one of $\{\frac{1}{2}t^2; t=1, 2, \ldots\}$. Using the relations

$$\alpha_k\geq 0; \quad \alpha_k\leq k-1; \quad \alpha_k\leq x_k-1; \quad \alpha_k\geq(x_k-1)-(n-k),$$

we obtain four inequalities which must be satisfied if $(k, x_k)$ is to be a zero pair. These are

$$k+x_k\geq\tfrac{1}{2}(n+3); \qquad k\geq\tfrac{1}{4}(2k+2x_k-n+1);$$

$$x_k\geq\tfrac{1}{4}(2k+2x_k-n+1); \qquad k+x_k\leq\tfrac{1}{2}(3n+1).$$

In this way we get a pictorial description of the pairs $(k, x_k)$ which can be a zero pair. These pairs lie in the closed square with vertices $(1, \frac{1}{2}(n+1))$, $(\frac{1}{2}(n+1), 1)$, $(\frac{1}{2}(n+1), n)$, $(n, \frac{1}{2}(n+1))$.

## 3.

We note that $(1, x_1)$, $(n, x_n)$ cannot both be a zero pair since $x_1=x_n=$

$\frac{1}{2}(n+1)$ is not admissible. Thus

$$\sum_{k=1,n} 2\{\alpha_k + \tfrac{1}{4}(n+3-2k-2x_k)\}^2 = \Sigma^{(1)} \geqslant \tfrac{1}{2}.$$

Consider next $(k, x_k)$ for $k = 1, 2, (n-1), n$ and write

$$\sum_{k=1,2,(n-1),n} 2\{\alpha_k + \tfrac{1}{4}(n+3-2k-2x_k)\}^2 = \Sigma^{(2)}.$$

It is a straightforward argument to show that $\Sigma^{(2)} \geqslant \frac{2}{2}$ and that equality is possible only for the four sets

$$\{(1, \tfrac{1}{2}(n-1)), (2, \tfrac{1}{2}(n-3)), (n-1, \tfrac{1}{2}(n+3)), (n, \tfrac{1}{2}(n+1))\},$$

$$\{(1, \tfrac{1}{2}(n+1)), (2, \tfrac{1}{2}(n-1)), (n-1, \tfrac{1}{2}(n+5)), (n, \tfrac{1}{2}(n+3))\},$$

$$\{(1, \tfrac{1}{2}(n+3)), (2, \tfrac{1}{2}(n+5)), (n-1, \tfrac{1}{2}(n-1)), (n, \tfrac{1}{2}(n+1))\},$$

$$\{(1, \tfrac{1}{2}(n+1)), (2, \tfrac{1}{2}(n+3)), (n-1, \tfrac{1}{2}(n-3)), (n, \tfrac{1}{2}(n-1))\}.$$

Let $u$ be an integer in $[3, \frac{1}{2}(n-1))$ and for $t = 1, 2, \ldots, u$ we assume that $\Sigma^{(t)} \geqslant \frac{1}{2}t$, where

$$\Sigma^{(t)} = \sum_{\substack{k=1,2,\ldots,t,\\ (n-t+1),\ldots,n}} 2\{\alpha_k + \tfrac{1}{4}(n+3-2k-2x_k)\}^2$$

and that equality is possible only for the four sets

$$\{(1, \tfrac{1}{2}(n-1)), (2, \tfrac{1}{2}(n-3)), \ldots, (t, \tfrac{1}{2}(n+1-2t)),$$
$$(n-t+1, \tfrac{1}{2}(n-1+2t)), \ldots, (n-1, \tfrac{1}{2}(n+3)), (n, \tfrac{1}{2}(n+1))\},$$

$$\{(1, \tfrac{1}{2}(n+1)), (2, \tfrac{1}{2}(n-1)), \ldots, (t, \tfrac{1}{2}(n+3-2t)),$$
$$(n-t+1, \tfrac{1}{2}(n+1+2t)), \ldots, (n-1, \tfrac{1}{2}(n+5)), (n, \tfrac{1}{2}(n+3))\},$$

$$\{(1, \tfrac{1}{2}(n+3)), (2, \tfrac{1}{2}(n+5)), \ldots, (t, \tfrac{1}{2}(n+1+2t)),$$
$$(n-t+1, \tfrac{1}{2}(n+3-2t)), \ldots, (n-1, \tfrac{1}{2}(n-1)), (n, \tfrac{1}{2}(n+1))\},$$

$$\{(1, \tfrac{1}{2}(n+1)), (2, \tfrac{1}{2}(n+3)), \ldots, (t, \tfrac{1}{2}(n-1+2t)),$$
$$(n-t+1, \tfrac{1}{2}(n+1-2t)), \ldots, (n-1, \tfrac{1}{2}(n-3)), (n, \tfrac{1}{2}(n-1))\},$$

and show that the statements hold for $t = u + 1$.

If $\Sigma^{(u)} > \frac{1}{2}u$, then $\Sigma^{(u+1)} \geqslant \frac{1}{2}(u+1)$ since any non-zero term contributes at least $\frac{1}{2}$.

If $\Sigma^{(u)} = \frac{1}{2}u$, then we have one of the four given sets, say, the first. Then there is no value $x_{u+1}$ which makes $(u+1, x_{u+1})$ a zero pair and so this pair contributes at least $\frac{1}{2}$ to $\Sigma^{(u+1)}$. Note that the pair $(n-u, \frac{1}{2}(n+1+2u))$ is a zero pair. So $\Sigma^{(u+1)} \geqslant \frac{1}{2}(u+1)$ in this case and similarly for the other three possibilities.

It remains to demonstrate that $\Sigma^{(u+1)} = \frac{1}{2}(u+1)$ implies one of the four specified configurations. If neither pair $(k, x_k)$ for $k = u+1, n-u$, is a zero pair, then $\Sigma^{(u+1)} \geqslant \Sigma^{(u)} + \frac{1}{2} + \frac{1}{2} > \frac{1}{2}(u+1)$, and if just one is a zero pair, then $\Sigma^{(u+1)} = \frac{1}{2}(u+1) \Rightarrow \Sigma^{(u)} = \frac{1}{2}u$ and we have one of the four situations just discussed. The proof is completed by showing that $(k, x_k)$ being a zero pair for $k = u+1$ and $k = n-u$ implies $\Sigma^{(u+1)} > \frac{1}{2}(u+1)$.

The reasoning of Section 2 gives $\frac{1}{2}(n+1) - u \leqslant x_{u+1}, x_{n-u} \leqslant \frac{1}{2}(n+1)+u$ and so at least one value, $\mu$, from $\{1, 2, \ldots, u, n-u+1, \ldots, n\}$ is such that $x_\mu \notin [\frac{1}{2}(n+1) - u, \frac{1}{2}(n+1) + u]$. There are four possibilities,

(i) $\mu \leqslant u$, $x_\mu < \frac{1}{2}(n+1)-u$;
(ii) $\mu \leqslant u$, $x_\mu > \frac{1}{2}(n+1)+u$;
(iii) $\mu \geqslant n-u+1$, $x_\mu < \frac{1}{2}(n+1)-u$; and
(iv) $\mu \geqslant n-u+1$, $x_\mu > \frac{1}{2}(n+1)+u$.

Consider (i). We have $\frac{1}{4}(n+3-2\mu-2x_\mu) \geqslant 1 + \frac{1}{2}(u-\mu)$ and using $\alpha_\mu \geqslant 0$ and the induction hypothesis for $t = \mu - 1$ we find

$$\Sigma^{(u+1)} - \tfrac{1}{2}(u+1) \geqslant \Sigma^{(\mu-1)} + 2\{1 + \tfrac{1}{2}(u-\mu)\}^2 - \tfrac{1}{2}(u+1)$$

$$\geqslant 1 + \tfrac{3}{2}(u-\mu) + \tfrac{1}{2}(u-\mu)^2 \geqslant 1 > 0,$$

as required.

Next, possibility (ii). Here $\mu + x_\mu \geqslant \mu + \frac{1}{2}(n+1) + u + 1$ and using $\alpha_\mu \leqslant \mu - 1$ we obtain

$$\alpha_\mu + \tfrac{1}{4}(n+3-2\mu-2x_\mu) \leqslant -(1 + \tfrac{1}{2}(u-\mu))$$

and the conclusion follows as before.

In case (iii) we employ $\alpha_\mu \leqslant x_\mu - 1$. Then

$$\alpha_\mu + \tfrac{1}{4}(n+3-2\mu-2x_\mu) = \tfrac{1}{2}x_\mu - \tfrac{1}{2}\mu - \tfrac{1}{4} + \tfrac{1}{4}n$$

$$\leqslant -\tfrac{1}{2}(1+\mu+u-n) \leqslant -1,$$

and hence

$$\Sigma^{(u+1)} - \tfrac{1}{2}(u+1) \geqslant \tfrac{1}{2}(\mu-1) + \tfrac{1}{2}(1+\mu+u-n)^2 - \tfrac{1}{2}(u+1)$$

$$= \tfrac{1}{2}(\mu-u-2) + \tfrac{1}{2}(1+\mu+u-n)^2 > 1.$$

Finally, in (iv) we use $\alpha_\mu \geqslant (x_\mu - 1) - (n-\mu)$ which leads to

$$\alpha_\mu + \tfrac{1}{4}(n+3-2\mu-2x_\mu) \geqslant \tfrac{1}{2}(1+\mu+u-n)$$

and $\Sigma^{(u+1)} > \frac{1}{2}(u+1)$ follows as in (iii).

This justifies our induction hypothesis up to $t = \frac{1}{2}(n-1)$ and by taking this value we deduce (4) and hence (2). The cases of equality have been established for $n$ odd, and $n$ even can be dealt with similarly. This completes the proof of the theorem.

## Reference

[1] H. Burkill and L. Mirsky, Monotonicity, J. Math. Anal. Appl. 41 (1973) 391–410.